AFRL-AFOSR-VA-TR-2015-0232

Geometric Structure-Preserving Discretization Schemes for Nonlinear Elasticity

Arash Yavari
GEORGIA TECH RESEARCH CORPORATION

08/13/2015
Final Report

DISTRIBUTION A: Distribution approved for public release.

Air Force Research Laboratory
AF Office Of Scientific Research (AFOSR)/ RTA
Arlington, Virginia 22203
Air Force Materiel Command

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From - To)*
14-08-2015	Final Performance	01-06-2012 to 31-05-2015

4. TITLE AND SUBTITLE
Geometric Structure-Preserving Discretization Schemes for Nonlinear Elasticity

5a. CONTRACT NUMBER

5b. GRANT NUMBER
FA9550-12-1-0290

5c. PROGRAM ELEMENT NUMBER
61102F

6. AUTHOR(S)
Arash Yavari

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
GEORGIA TECH RESEARCH CORPORATION
505 10TH ST NW
ATLANTA, GA 30318-5775 US

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
AF Office of Scientific Research
875 N. Randolph St. Room 3112
Arlington, VA 22203

10. SPONSOR/MONITOR'S ACRONYM(S)
AFRL/AFOSR RTA

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
A DISTRIBUTION UNLIMITED: PB Public Release

13. SUPPLEMENTARY NOTES

14. ABSTRACT
We introduced a smooth complex for nonlinear elasticity that can be considered as the tensorial analogue of the standard grad-curl-div complex. This mathematical structure simultaneously describes the kinematics and the kinetics of large deformations. The relation between this complex and the de Rham complex allows one to readily derive the necessary and sufficient conditions for the compatibility of displacement gradient and the existence of stress functions on non-contractible bodies. The main application of the nonlinear elasticity complex is in developing mixed finite element methods for large deformations, which will be pursued in a future project. To this end, the smooth complex should be extended to also include less smooth tensors. We introduced this extension by using the so-called partly Sobolev spaces. The result is a Hilbert complex involving second-order tensors on flat compact manifolds with boundary. We then used the general framework of Hilbert complexes to write Hodge-type and Helmholtz-type orthogonal decompositions for second-order tensors. As some applications of these decompositions in continuum mechanics, one can study the strain compatibility equations of nonlinear elasticity in the presence of Dirichlet boundary conditions.

15. SUBJECT TERMS
geometric theory for nonlinear elasticity, discrete exterior calculus

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF

Standard Form 298 (Rev. 8/98)
Prescribed by ANSI Std. Z39.18

DISTRIBUTION A: Distribution approved for public release

a. REPORT	b. ABSTRACT	c. THIS PAGE		PAGES	19a. NAME OF RESPONSIBLE PERSON
Unclassified	Unclassified	Unclassified	UU		Arash Yavari
					19b. TELEPHONE NUMBER *(Include area code)*
					404-894-2436

Final Report for
GEOMETRIC STRUCTURE-PRESERVING DISCRETIZATION SCHEMES FOR ELASTICITY

AFOSR GRANT NUMBER FA9550-12-1-0290

Arash Yavari
Department of Civil and Environmental Engineering
Georgia Institute of Technology, Atlanta

1 Summary

We introduced a smooth complex for nonlinear elasticity that can be considered as the tensorial analogue of the standard grad-curl-div complex. This mathematical structure simultaneously describes the kinematics and the kinetics of large deformations. The relation between this complex and the de Rham complex allows one to readily derive the necessary and sufficient conditions for the compatibility of displacement gradient and the existence of stress functions on non-contractible bodies. The main application of the nonlinear elasticity complex is in developing mixed finite element methods for large deformations, which will be pursued in a future project. To this end, the smooth complex should be extended to also include less smooth tensors. We introduced this extension by using the so-called partly Sobolev spaces. The result is a Hilbert complex involving second-order tensors on flat compact manifolds with boundary. We then used the general framework of Hilbert complexes to write Hodge-type and Helmholtz-type orthogonal decompositions for second-order tensors. As some applications of these decompositions in continuum mechanics, one can study the strain compatibility equations of nonlinear elasticity in the presence of Dirichlet boundary conditions.

For developing finite element methods for nonlinear elasticity using the above Hilbert complex, we first discretize this complex by means of appropriate finite element spaces and drive stable mixed finite element methods for the associated Laplacian. We use the general theory for approximation of Hilbert complexes and the finite element exterior calculus and introduce some stable mixed finite element methods for the Laplacian of second-order tensors with appropriate mixed boundary conditions. To this end, we introduce new finite elements for second-order tensors that are the tensorial analogues of some standard finite elements for vector fields. One important feature of the finite element spaces generated by these finite elements is that they respect the global topology of the underlying domains in the sense that they reproduce some topological properties of the domains regardless of the refinement level of meshes.

2 Nonlinear Elasticity Complex

Let $\mathcal{B} \subset \mathbb{R}^3$ be an open subset and suppose $\{X^I\}$ is the Cartesian coordinates on $\mathcal{B}$. We equip $\mathcal{B}$ with metric $\boldsymbol{G}$, which is the Euclidean metric of $\mathbb{R}^3$. The gradient of vector fields and the curl and the

divergence of $\binom{2}{0}$-tensors are defined as

$$
\begin{aligned}
\mathbf{grad} &: \mathfrak{X}(\mathcal{B}) \to \Gamma(\otimes^2 T\mathcal{B}), & (\mathbf{grad}\,\boldsymbol{Y})^{IJ} &= Y^I{}_{,J}, \\
\mathbf{curl} &: \Gamma(\otimes^2 T\mathcal{B}) \to \Gamma(\otimes^2 T\mathcal{B}), & (\mathbf{curl}\,\boldsymbol{T})^{IJ} &= \varepsilon_{IKL} T^{JL}{}_{,K}, \\
\mathbf{div} &: \Gamma(\otimes^2 T\mathcal{B}) \to \mathfrak{X}(\mathcal{B}), & (\mathbf{div}\,\boldsymbol{T})^{I} &= T^{IJ}{}_{,J},
\end{aligned}
$$

where "$_{,J}$" indicates $\partial/\partial X^J$. We also define the operator

$$
\mathbf{curl}^{\mathsf{T}} : \Gamma(\otimes^2 T\mathcal{B}) \to \Gamma(\otimes^2 T\mathcal{B}),\ \ (\mathbf{curl}^{\mathsf{T}}\boldsymbol{T})^{IJ} = (\mathbf{curl}\,\boldsymbol{T})^{JI}.
$$

It is straightforward to show that $\mathbf{curl}^{\mathsf{T}} \circ \mathbf{grad} = 0$, and $\mathbf{div} \circ \mathbf{curl}^{\mathsf{T}} = 0$. Thus, we obtain the following complex

$$
0 \longrightarrow \mathfrak{X}(\mathcal{B}) \xrightarrow{\mathbf{grad}} \Gamma(\otimes^2 T\mathcal{B}) \xrightarrow{\mathbf{curl}^{\mathsf{T}}} \Gamma(\otimes^2 T\mathcal{B}) \xrightarrow{\mathbf{div}} \mathfrak{X}(\mathcal{B}) \longrightarrow 0, \tag{2.1}
$$

that, due to its resemblance with the gcd complex (i.e. the standard grad-curl-div complex of vector fields), is called the **gcd** complex. Interestingly, similar to the gcd complex, useful properties of the **gcd** complex also follow from the de Rham complex. This can be described via the $\mathbb{R}^3$-valued de Rham complex as follows. Let $d : \Omega^k(\mathcal{B}) \to \Omega^{k+1}(\mathcal{B})$ be the standard exterior derivative given by

$$
(d\boldsymbol{\beta})_{I_0 \cdots I_k} = \sum_{i=0}^{k} (-1)^i \beta_{I_0 \cdots \widehat{I_i} \cdots I_k, I_i},
$$

where the hat over an index implies the elimination of that index. Any $\boldsymbol{\alpha} \in \Omega^k(\mathcal{B}; \mathbb{R}^3)$ can be considered as $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2, \boldsymbol{\alpha}^3)$, with $\boldsymbol{\alpha}^i \in \Omega^k(\mathcal{B})$, $i = 1,2,3$. One can define the exterior derivative $\boldsymbol{d} : \Omega^k(\mathcal{B}; \mathbb{R}^3) \to \Omega^{k+1}(\mathcal{B}; \mathbb{R}^3)$ by $\boldsymbol{d}\boldsymbol{\alpha} = (d\boldsymbol{\alpha}^1, d\boldsymbol{\alpha}^2, d\boldsymbol{\alpha}^3)$. Since $d \circ d = 0$, we also conclude that $\boldsymbol{d} \circ \boldsymbol{d} = 0$, which leads to the $\mathbb{R}^3$-valued de Rham complex $\big(\Omega(\mathcal{B}; \mathbb{R}^3), \boldsymbol{d}\big)$. Given $\boldsymbol{\alpha} \in \Omega^k(\mathcal{B}; \mathbb{R}^3)$, let $[\boldsymbol{\alpha}]^i{}_{I_1 \cdots I_k}$ denote the components of $\boldsymbol{\alpha}^i \in \Omega^k(\mathcal{B})$. By using the global orthonormal coordinate system $\{X^I\}$, one can define the following isomorphisms

$$
\begin{aligned}
\boldsymbol{\imath}_0 &: \mathfrak{X}(\mathcal{B}) \to \Omega^0(\mathcal{B}; \mathbb{R}^3), & [\boldsymbol{\imath}_0(\boldsymbol{Y})]^i &= \delta_{iI} Y^I, \\
\boldsymbol{\imath}_1 &: \Gamma(\otimes^2 T\mathcal{B}) \to \Omega^1(\mathcal{B}; \mathbb{R}^3), & [\boldsymbol{\imath}_1(\boldsymbol{T})]^i{}_J &= \delta_{iI} T^{IJ}, \\
\boldsymbol{\imath}_2 &: \Gamma(\otimes^2 T\mathcal{B}) \to \Omega^2(\mathcal{B}; \mathbb{R}^3), & [\boldsymbol{\imath}_2(\boldsymbol{T})]^i{}_{JK} &= \delta_{iI} \varepsilon_{JKL} T^{IL}, \\
\boldsymbol{\imath}_3 &: \mathfrak{X}(\mathcal{B}) \to \Omega^3(\mathcal{B}; \mathbb{R}^3), & [\boldsymbol{\imath}_3(\boldsymbol{Y})]^i{}_{123} &= \delta_{iI} Y^I,
\end{aligned}
$$

where δ_{iI} is the Kronecker delta. Let $\boldsymbol{T}^{\mathsf{T}}$ be the transpose of $\boldsymbol{T}$, i.e. $\big(\boldsymbol{T}^{\mathsf{T}}\big)^{IJ} = T^{JI}$, and let $\{\mathbf{E}_I\}$ be the standard basis of $\mathbb{R}^3$. For $\boldsymbol{T} \in \Gamma(\otimes^2 T\mathcal{B})$, we define $\overrightarrow{\boldsymbol{T}}_{\mathbf{N}}$ to be the traction of $\boldsymbol{T}^{\mathsf{T}}$ in the direction of unit vector $\mathbf{N} = N^I \mathbf{E}_I \in \mathbb{S}^2$, where $\mathbb{S}^2 \subset \mathbb{R}^3$ is the unit 2-sphere. Thus, $\overrightarrow{\boldsymbol{T}}_{\mathbf{N}} = N^I T^{IJ} \mathbf{E}_J$. One can write

$$
\boldsymbol{\imath}_k(\boldsymbol{T}) = \Big(\imath_k\big(\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_1}\big), \imath_k\big(\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_2}\big), \imath_k\big(\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_3}\big) \Big),\ \ k = 1,2. \tag{2.2}
$$

It is easy to show that

$$
\boldsymbol{\imath}_1 \circ \mathbf{grad} = \boldsymbol{d} \circ \boldsymbol{\imath}_0,\ \ \boldsymbol{\imath}_2 \circ \mathbf{curl}^{\mathsf{T}} = \boldsymbol{d} \circ \boldsymbol{\imath}_1,\ \ \boldsymbol{\imath}_3 \circ \mathbf{div} = \boldsymbol{d} \circ \boldsymbol{\imath}_2.
$$

Therefore, the following diagram commutes for the **gcd** complex.

$$
\begin{tikzcd}
0 \arrow[r] & \mathfrak{X}(\mathcal{B}) \arrow[r, "\mathbf{grad}"] \arrow[d, "\imath_0"] & \Gamma(\otimes^2 T\mathcal{B}) \arrow[r, "\mathbf{curl}^{\mathsf{T}}"] \arrow[d, "\imath_1"] & \Gamma(\otimes^2 T\mathcal{B}) \arrow[r, "\mathbf{div}"] \arrow[d, "\imath_2"] & \mathfrak{X}(\mathcal{B}) \arrow[r] \arrow[d, "\imath_3"] & 0 \\
0 \arrow[r] & \Omega^0(\mathcal{B};\mathbb{R}^3) \arrow[r, "\boldsymbol{d}"] & \Omega^1(\mathcal{B};\mathbb{R}^3) \arrow[r, "\boldsymbol{d}"] & \Omega^2(\mathcal{B};\mathbb{R}^3) \arrow[r, "\boldsymbol{d}"] & \Omega^3(\mathcal{B};\mathbb{R}^3) \arrow[r] & 0
\end{tikzcd}
\tag{2.3}
$$

The contraction $\langle \boldsymbol{T}, \boldsymbol{Y} \rangle$ of $\boldsymbol{T} \in \Gamma(\otimes^2 T\mathcal{B})$ and $\boldsymbol{Y} \in \mathfrak{X}(\mathcal{B})$ is a vector field that in the orthonormal coordinate system $\{X^I\}$ reads $\langle \boldsymbol{T}, \boldsymbol{Y} \rangle = T^{IJ} Y^J \mathbf{E}_I$. Clearly, if $\boldsymbol{N}_{\mathcal{C}}$ is the unit outward normal vector field of a closed surface $\mathcal{C} \subset \mathcal{B}$, then $\langle \boldsymbol{T}, \boldsymbol{N}_{\mathcal{C}} \rangle$ is the traction of $\boldsymbol{T}$ on $\mathcal{C}$. Suppose $\mathrm{H}^k_{\mathbf{gcd}}(\mathcal{B})$ is the k-th cohomology group of the **gcd** complex. Diagram (2.3) implies that $\imath_k$ also induces the isomorphism $\mathrm{H}^k_{\mathbf{gcd}}(\mathcal{B}) \approx \oplus_{i=1}^3 \mathrm{H}^k_{dR}(\mathcal{B})$ between the cohomology groups. Using this fact, one can prove the following theorem.

Theorem 2.1. *An arbitrary tensor $\boldsymbol{T} \in \Gamma(\otimes^2 T\mathcal{B})$ is the gradient of a vector field if and only if*

$$
\mathbf{curl}^{\mathsf{T}} \boldsymbol{T} = 0, \;\; \textit{and} \qquad \int_{\ell} \langle \boldsymbol{T}, \boldsymbol{t}_{\ell} \rangle dS = 0, \;\; \forall \ell \subset \mathcal{B}, \tag{2.4}
$$

where ℓ is an arbitrary closed curve in $\mathcal{B}$ and $\mathbf{t}_{\ell}$ is the unit tangent vector field along ℓ.

Similarly, one can derive the necessary and sufficient conditions for the existence of a potential for $\boldsymbol{T}$ induced by $\mathbf{curl}^{\mathsf{T}}$. The upshot is the following theorem.

Theorem 2.2. *Given $\boldsymbol{T} \in \Gamma(\otimes^2 T\mathcal{B})$, there exists $\boldsymbol{W} \in \Gamma(\otimes^2 T\mathcal{B})$ such that $\boldsymbol{T} = \mathbf{curl}^{\mathsf{T}} \boldsymbol{W}$, if and only if*

$$
\mathbf{div}\, \boldsymbol{T} = 0, \;\; \textit{and} \qquad \int_{\mathcal{C}} \langle \boldsymbol{T}, \boldsymbol{N}_{\mathcal{C}} \rangle dA = 0, \;\; \forall \mathcal{C} \subset \mathcal{B}, \tag{2.5}
$$

where $\mathcal{C}$ is an arbitrary closed surface in $\mathcal{B}$ and $\mathbf{N}_{\mathcal{C}}$ is its unit outward normal vector field.

We can also write an analogue of the **gcd** complex for two-point tensors. Let $\mathcal{S} = \mathbb{R}^3$ with coordinate system $\{x^i\}$, which is the Cartesian coordinates of $\mathbb{R}^3$. Suppose $\varphi : \mathcal{B} \to \mathcal{S}$ is a smooth mapping and let $T_X\varphi(\mathcal{B}) := T_{\varphi(X)}\mathcal{S}$. Note that although φ is not necessarily an embedding, the dimension of $T_X\varphi(\mathcal{B})$ is always equal to $\dim \mathcal{S}$. We can define the following operators for two-point tensors that belong to $\Gamma(T\varphi(\mathcal{B}))$ and $\Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B})$:

$$
\begin{aligned}
\mathbf{Grad} &: \Gamma(T\varphi(\mathcal{B})) \to \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}), & (\mathbf{Grad}\, \boldsymbol{U})^{iI} &= U^i{}_{,I}, \\
\mathbf{Curl}^{\mathsf{T}} &: \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \to \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}), & (\mathbf{Curl}^{\mathsf{T}} \boldsymbol{F})^{iI} &= \varepsilon_{IKL} F^{iL}{}_{,K}, \\
\mathbf{Div} &: \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \to \Gamma(T\varphi(\mathcal{B})), & (\mathbf{Div}\, \boldsymbol{F})^i &= F^{iI}{}_{,I}.
\end{aligned}
$$

We have $\mathbf{Curl}^{\mathsf{T}} \circ \mathbf{Grad} = 0$, and $\mathbf{Div} \circ \mathbf{Curl}^{\mathsf{T}} = 0$. Thus, the **GCD** complex, which we also call the nonlinear elasticity complex, can be written as:

$$
0 \longrightarrow \Gamma(T\varphi(\mathcal{B})) \xrightarrow{\mathbf{Grad}} \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \xrightarrow{\mathbf{Curl}^{\mathsf{T}}} \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \xrightarrow{\mathbf{Div}} \Gamma(T\varphi(\mathcal{B})) \longrightarrow 0.
$$

By using the following isomorphisms

$$
\begin{aligned}
&\boldsymbol{I}_0 : \Gamma(T\varphi(\mathcal{B})) \to \Omega^0(\mathcal{B};\mathbb{R}^3), && [\boldsymbol{I}_0(\boldsymbol{U})]^i = U^i,\\
&\boldsymbol{I}_1 : \Gamma(T\varphi(\mathcal{B})\otimes T\mathcal{B}) \to \Omega^1(\mathcal{B};\mathbb{R}^3), && [\boldsymbol{I}_1(\boldsymbol{F})]^i{}_J = F^{iJ},\\
&\boldsymbol{I}_2 : \Gamma(T\varphi(\mathcal{B})\otimes T\mathcal{B}) \to \Omega^2(\mathcal{B};\mathbb{R}^3), && [\boldsymbol{I}_2(\boldsymbol{F})]^i{}_{JK} = \varepsilon_{JKL}F^{iL},\\
&\boldsymbol{I}_3 : \Gamma(T\varphi(\mathcal{B})) \to \Omega^3(\mathcal{B};\mathbb{R}^3), && [\boldsymbol{I}_3(\boldsymbol{U})]^i{}_{123} = U^i,
\end{aligned}
$$

one concludes that the following diagram commutes.

$$
\begin{tikzcd}
0 \arrow[r] & \Gamma(T\varphi(\mathcal{B})) \arrow[r, "\mathbf{Grad}"] \arrow[d, "\boldsymbol{I}_0"] & \Gamma(T\varphi(\mathcal{B})\otimes T\mathcal{B}) \arrow[r, "\mathbf{Curl}^{\mathsf{T}}"] \arrow[d, "\boldsymbol{I}_1"] & \Gamma(T\varphi(\mathcal{B})\otimes T\mathcal{B}) \arrow[r, "\mathbf{Div}"] \arrow[d, "\boldsymbol{I}_2"] & \Gamma(T\varphi(\mathcal{B})) \arrow[r] \arrow[d, "\boldsymbol{I}_3"] & 0\\
0 \arrow[r] & \Omega^0(\mathcal{B};\mathbb{R}^3) \arrow[r, "\boldsymbol{d}"] & \Omega^1(\mathcal{B};\mathbb{R}^3) \arrow[r, "\boldsymbol{d}"] & \Omega^2(\mathcal{B};\mathbb{R}^3) \arrow[r, "\boldsymbol{d}"] & \Omega^3(\mathcal{B};\mathbb{R}^3) \arrow[r] & 0
\end{tikzcd}
$$

The above isomorphisms also induce an isomorphism $\mathrm{H}^k_{\mathsf{GCD}}(\mathcal{B}) \approx \oplus_{i=1}^3 \mathrm{H}^k_{dR}(\mathcal{B})$, where $\mathrm{H}^k_{\mathsf{GCD}}(\mathcal{B})$ is the k-th cohomology group of the **GCD** complex. Let $\{\mathbf{E}_I\}$ and $\{\mathbf{e}_i\}$ be two copies of the standard basis of $\mathbb{R}^3$. For $\boldsymbol{F} \in \Gamma(T\varphi(\mathcal{B})\otimes T\mathcal{B})$, and $\mathbf{n} = n^i\mathbf{e}_i \in \mathcal{S}^2$, let $\overrightarrow{\boldsymbol{F}}_{\mathbf{n}} = n^iF^{iJ}\mathbf{E}_J \in \mathfrak{X}(\mathcal{B})$. Then, one can write

$$\boldsymbol{I}_k(\boldsymbol{F}) = \left(\imath_k\Big(\overrightarrow{\boldsymbol{F}}_{\mathbf{e}_1}\Big), \imath_k\Big(\overrightarrow{\boldsymbol{F}}_{\mathbf{e}_2}\Big), \imath_k\Big(\overrightarrow{\boldsymbol{F}}_{\mathbf{e}_3}\Big)\right),\ k = 1, 2.$$

Let $\langle\boldsymbol{F}, \boldsymbol{Y}\rangle := F^{iI}Y^I\mathbf{e}_i$. The above relations for the **GCD** complex allow us to obtain the following results that can be proved similarly to Theorems 2.1 and 2.2.

Theorem 2.3. *Given $\boldsymbol{F} \in \Gamma(T\varphi(\mathcal{B})\otimes T\mathcal{B})$, there exists $\boldsymbol{U} \in \Gamma(T\varphi(\mathcal{B}))$ such that $\boldsymbol{F} = \mathbf{Grad}\,\boldsymbol{U}$, if and only if*

$$\mathbf{Curl}^{\mathsf{T}}\boldsymbol{F} = 0,\ \textit{and} \qquad \int_\ell \langle\boldsymbol{F}, \boldsymbol{t}_\ell\rangle dS = 0,\ \forall \ell \subset \mathcal{B}.$$

Moreover, there exists $\boldsymbol{\Psi} \in \Gamma(T\varphi(\mathcal{B})\otimes T\mathcal{B})$ such that $\boldsymbol{F} = \mathbf{Curl}^{\mathsf{T}}\boldsymbol{\Psi}$, if and only if

$$\mathbf{Div}\,\boldsymbol{F} = 0,\ \textit{and} \qquad \int_{\mathcal{C}} \langle\boldsymbol{F}, \boldsymbol{N}_{\mathcal{C}}\rangle dA = 0,\ \forall \mathcal{C} \subset \mathcal{B}.$$

3 Hilbert Complexes and Orthogonal Decompositions

Consider the following linear subspaces of $\Gamma(T\varphi(\bar{\mathcal{B}}))$ and $\Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})$:

$$
\begin{aligned}
\Gamma_j(T\varphi(\bar{\mathcal{B}})) &:= \left\{\boldsymbol{U} \in \Gamma(T\varphi(\bar{\mathcal{B}})) : \boldsymbol{U}|_{\partial_j\bar{\mathcal{B}}} = 0\right\},\\
\Gamma_{\mathrm{n}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) &:= \left\{\boldsymbol{F} \in \Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) : \overrightarrow{\boldsymbol{F}}_{\mathbf{e}_i} \perp \partial_j\bar{\mathcal{B}},\ i = 1, \ldots, n\right\},\\
\Gamma_{\mathrm{t}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) &:= \left\{\boldsymbol{F} \in \Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) : \overrightarrow{\boldsymbol{F}}_{\mathbf{e}_i} \| \partial_j\bar{\mathcal{B}},\ i = 1, \ldots, n\right\}.
\end{aligned}
$$

The operators **Grad**, $\mathbf{Curl}^{\mathsf{T}}$, and **Div** can be restricted to the above subspaces which allows one to impose boundary conditions on the **GCD** complex. The upshot is the following commutative diagrams.

$$\begin{array}{ccccccccc}
0 \longrightarrow \Gamma_j(T\varphi(\bar{\mathcal{B}})) & \xrightarrow{\mathbf{Grad}_j} & \Gamma_{\mathrm{n}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{Curl}_j^{\mathsf{T}}} & \Gamma_{\mathrm{t}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{Div}_j} & \Gamma(T\varphi(\bar{\mathcal{B}})) \longrightarrow 0 \\
\downarrow \boldsymbol{I}_0 & & \downarrow \boldsymbol{I}_1 & & \downarrow \boldsymbol{I}_2 & & \downarrow \boldsymbol{I}_3 \\
0 \longrightarrow \Omega^0_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} & \Omega^1_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} & \Omega^2_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} & \Omega^3_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^3) \longrightarrow 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 \longleftarrow \Gamma(T\varphi(\bar{\mathcal{B}})) & \xleftarrow{\mathbf{Div}_j} & \Gamma_{\mathrm{t}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) & \xleftarrow{\mathbf{Curl}_j^{\mathsf{T}}} & \Gamma_{\mathrm{n}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) & \xleftarrow{\mathbf{Grad}_j} & \Gamma_j(T\varphi(\bar{\mathcal{B}})) \longleftarrow 0 \\
\downarrow -\boldsymbol{I}_0 & & \downarrow \boldsymbol{I}_1 & & \downarrow \boldsymbol{I}_2 & & \downarrow -\boldsymbol{I}_3 \\
0 \longleftarrow \Omega^0_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} & \Omega^1_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} & \Omega^2_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} & \Omega^3_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^3) \longleftarrow 0
\end{array}$$

The isomorphisms $\boldsymbol{I}_0,\cdots,\boldsymbol{I}_3$ are L^2-isometries. The Hilbert spaces $L^2\Gamma(T\varphi(\bar{\mathcal{B}}))$, $H^1\Gamma_j(T\varphi(\bar{\mathcal{B}}))$, $H^{\mathbf{C}}\Gamma_{\mathrm{n}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})$, and $H^{\mathbf{D}}\Gamma_{\mathrm{t}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})$ are the completions of $\big(\Gamma(T\varphi(\bar{\mathcal{B}})),\langle\!\langle,\rangle\!\rangle_{L^2}\big)$, $\big(\Gamma_j(T\varphi(\bar{\mathcal{B}})),\langle\!\langle,\rangle\!\rangle_{H^1}\big)$, $\big(\Gamma_{\mathrm{n}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}),\langle\!\langle,\rangle\!\rangle_{H^{\mathbf{C}}}\big)$, and $\big(\Gamma_{\mathrm{t}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}),\langle\!\langle,\rangle\!\rangle_{H^{\mathbf{D}}}\big)$, respectively. These Hilbert spaces allow one to write the following Hilbert complex for two-point tensors:

$$\begin{aligned} 0 \longrightarrow H^1\Gamma_1(T\varphi(\bar{\mathcal{B}})) \xrightarrow{\mathbf{Grad}_1} H^{\mathbf{C}}\Gamma_{\mathrm{n}_1}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) \xrightarrow{\mathbf{Curl}_1^{\mathsf{T}}} \\ H^{\mathbf{D}}\Gamma_{\mathrm{t}_1}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) \xrightarrow{\mathbf{Div}_1} L^2\Gamma(T\varphi(\bar{\mathcal{B}})) \longrightarrow 0 \end{aligned} \tag{3.1}$$

The dual of this Hilbert complex reads:

$$\begin{aligned} \xleftarrow{\mathbf{Curl}_2^{\mathsf{T}}} H^{\mathbf{C}}\Gamma_{\mathrm{n}_2}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) \xleftarrow{-\mathbf{Grad}_2} H^1\Gamma_2(T\varphi(\bar{\mathcal{B}})) \longleftarrow 0 \\ 0 \longleftarrow L^2\Gamma(T\varphi(\bar{\mathcal{B}})) \xleftarrow{-\mathbf{Div}_2} H^{\mathbf{D}}\Gamma_{\mathrm{t}_2}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) \end{aligned} \tag{3.2}$$

The complex (3.1) is isomorphic to $(\boldsymbol{H}^{\boldsymbol{d}}\Omega_{\mathrm{n}_1}(\bar{\mathcal{B}}),\boldsymbol{d}_{\mathrm{n}_1})$, and hence, it is Fredholm with $\mathsf{H}^k_{\mathbf{GCD}_1}(\bar{\mathcal{B}}) \approx \mathrm{H}^k_{\mathbf{GCD}_1}(\bar{\mathcal{B}}) \approx \oplus_{i=1}^3 \mathrm{H}^k_{dR}(\bar{\mathcal{B}},\partial_1\mathcal{B})$, where $\mathrm{H}^k_{\mathbf{GCD}_1}(\bar{\mathcal{B}})$ and $\mathsf{H}^k_{\mathbf{GCD}_1}(\bar{\mathcal{B}})$ are the k-th cohomologies of the smooth **GCD** complex (with boundary conditions on $\partial_1\bar{\mathcal{B}}$) and the Hilbert complex (3.1), respectively. Let $\mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) \coloneqq \ker \mathbf{Curl}_1^{\mathsf{T}} \cap \ker \mathbf{Div}_2$ be the kernel of the Laplacian L_φ associated to (3.1) and (3.2). Then, $\mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}})$ only consists of smooth harmonic two-point tensors and $\mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) \approx \mathsf{H}^1_{\mathbf{GCD}_1}(\bar{\mathcal{B}})$. One can show that:

Theorem 3.1. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^3$ be a smooth, compact 3-manifold with boundary and suppose $\varphi : \bar{\mathcal{B}} \to \mathbb{R}^3$ is a smooth mapping. The Hilbert complex (3.1) induces the following L^2-orthogonal decompositions: The Hodge decomposition*

$$\begin{aligned} &L^2\Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) = \\ &\mathbf{Grad}\big(H^1\Gamma_1(T\varphi(\bar{\mathcal{B}}))\big) \oplus \mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) \oplus \mathbf{Curl}^{\mathsf{T}}\big(H^{\mathbf{C}}\Gamma_{\mathrm{n}_2}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})\big), \end{aligned}$$

and, equivalently, the Helmholtz decompositions

$$\begin{aligned} L^2\Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) &= \mathbf{Grad}\big(H^1\Gamma_1(T\varphi(\bar{\mathcal{B}}))\big) \oplus \ker \mathbf{Div}_2 \\ &= \ker \mathbf{Curl}_1^{\mathsf{T}} \oplus \mathbf{Curl}^{\mathsf{T}}\big(H^{\mathbf{C}}\Gamma_{\mathrm{n}_2}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})\big), \end{aligned}$$

where

$$\begin{aligned}\ker \mathbf{Div}_2 &= \mathbf{Curl}^{\mathsf{T}}\big(H^{\mathbf{C}}\Gamma_{\mathrm{n}_2}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})\big) \oplus \mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}),\\ \ker \mathbf{Curl}^{\mathsf{T}}_1 &= \mathbf{Grad}\big(H^1\Gamma_1(T\varphi(\bar{\mathcal{B}}))\big) \oplus \mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}).\end{aligned}$$

If in addition a two-point tensor is of class $C^{r,\mu}$ (C^∞), then its components in the above decompositions are of class $C^{r,\mu}$ (C^∞) as well.

Corollary 3.2. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^3$ be a smooth, compact 3-manifold with boundary and suppose $\varphi : \bar{\mathcal{B}} \to \mathbb{R}^3$ is a smooth mapping. The necessary and sufficient conditions for the existence of a $\mathbf{Grad}_1$-potential for $\boldsymbol{F} \in L^2\Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})$ are*

$$\boldsymbol{F} \in H^{\mathbf{C}}\Gamma_{\mathrm{n}_1}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}),\ \mathbf{Curl}^{\mathsf{T}}\boldsymbol{F} = 0,\ \langle\!\langle \boldsymbol{F}, \boldsymbol{K} \rangle\!\rangle_{L^2} = 0,\ \forall \boldsymbol{K} \in \mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}).$$

Similarly, the necessary and sufficient conditions for the existence of a $\mathbf{Curl}^{\mathsf{T}}_2$-potential for $\boldsymbol{F}$ are

$$\boldsymbol{F} \in H^{\mathbf{D}}\Gamma_{\mathrm{t}_2}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}),\ \mathbf{Div}\,\boldsymbol{F} = 0,\ \langle\!\langle \boldsymbol{F}, \boldsymbol{K} \rangle\!\rangle_{L^2} = 0,\ \forall \boldsymbol{K} \in \mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}).$$

Similar results are also valid for 2-manifolds in $\mathbb{R}^2$.

4 Mixed formulations of the tensor Laplacian

By the (weak) 3D Laplacian for second-order tensors (or simply the 3D tensor Laplacian) with mixed boundary conditions, we mean the following boundary-value problem: Given $\boldsymbol{Q} \in L^2(\otimes^2 T\mathcal{B})$, find $\boldsymbol{T} \in L^2(\otimes^2 T\mathcal{B})$ such that

$$\begin{aligned}&\mathscr{L}(\boldsymbol{T}) = \boldsymbol{Q},\\ &\boldsymbol{T} \perp \mathcal{S}_1,\ (\mathbf{div}\,\boldsymbol{T})|_{\mathcal{S}_1} = 0, \quad \boldsymbol{T} \| \mathcal{S}_2,\ \big(\mathbf{curl}^{\mathsf{T}}\boldsymbol{T}\big) \perp \mathcal{S}_2,\end{aligned} \tag{4.1}$$

where $\mathscr{L} := \mathbf{curl}^{\mathsf{T}} \circ \mathbf{curl}^{\mathsf{T}} - \mathbf{grad} \circ \mathbf{div}$. The space of solutions of (4.1) for $\boldsymbol{Q} = 0$, is the space of *harmonic tensors* defined as

$$\mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2) := \ker \mathscr{L}_1 = \ker \mathbf{curl}^{\mathsf{T}}_{\mathcal{S}_1} \cap \ker \mathbf{div}_{\mathcal{S}_2}.$$

One can write

$$\dim \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2) = \dim \mathcal{H}^1(\mathcal{B}, \mathcal{S}_1) = 3b_1(\bar{\mathcal{B}}, \mathcal{S}_1). \tag{4.2}$$

Note that $\mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$ measures the non-uniqueness of solutions of (4.1) in the sense that if $\boldsymbol{T}$ is a solution of (4.1), then so is $\boldsymbol{T} + \hat{\boldsymbol{T}}$, $\forall \hat{\boldsymbol{T}} \in \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$. The integer $b_1(\bar{\mathcal{B}}, \mathcal{S}_1)$ depends on the topological properties of both $\mathcal{B}$ and $\mathcal{S}_1$. For example, if $\mathcal{B}$ is a 3D ball with a spherical hole, then $b_1(\bar{\mathcal{B}}, \varnothing) = 0$, and $b_1(\bar{\mathcal{B}}, \partial\mathcal{B}) = 1$. Therefore, (4.2) provides a connection between solutions of (4.1) and topological properties of $\mathcal{B}$ and $\mathcal{S}_1$.

Since $\mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$ is finite-dimensional, it is a closed subspace of $L^2(\otimes^2 T\mathcal{B})$, and therefore, using the orthogonal projection theorem, one can write $L^2(\otimes^2 T\mathcal{B}) = \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2) \oplus \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)^\perp$, where

$\mathcal{H}(\mathcal{B},\mathcal{S}_1,\mathcal{S}_2)^\perp$ is the orthogonal complement of $\mathcal{H}(\mathcal{B},\mathcal{S}_1,\mathcal{S}_2)$, i.e.

$$\mathcal{H}(\mathcal{B},\mathcal{S}_1,\mathcal{S}_2)^\perp := \left\{\boldsymbol{T} \in L^2(\otimes^2 T\mathcal{B}) : \langle\!\langle \boldsymbol{T},\boldsymbol{S}\rangle\!\rangle_{L^2\otimes^2} = 0,\ \forall \boldsymbol{S} \in \mathcal{H}(\mathcal{B},\mathcal{S}_1,\mathcal{S}_2)\right\}.$$

One can show that:

Theorem 4.1. *The boundary-value problem (4.1) admits a solution if and only if* $\boldsymbol{Q} \in \mathcal{H}(\mathcal{B},\mathcal{S}_1,\mathcal{S}_2)^\perp$.

For approximating solutions of the tensor Laplacians with mixed boundary conditions by means of Galerkin methods, we need suitable weak formulations for these boundary-value problems. The boundary-value problem (4.1) can be considered as a special case of the *abstract Hodge Laplacian* and therefore, one can define the following mixed formulation for (4.1): Given $\boldsymbol{Q} \in L^2(\otimes^2 T\mathcal{B})$, find $(\boldsymbol{U},\boldsymbol{T},\boldsymbol{P}) \in H^1(T\mathcal{B},\mathcal{S}_1) \times H^{\mathbf{c}}(\otimes^2 T\mathcal{B},\mathcal{S}_1) \times \mathcal{H}(\mathcal{B},\mathcal{S}_1,\mathcal{S}_2)$ such that

$$\begin{aligned}
\langle\!\langle \boldsymbol{U},\boldsymbol{\Upsilon}\rangle\!\rangle_{L^2T} - \langle\!\langle \boldsymbol{T},\mathbf{grad}\,\boldsymbol{\Upsilon}\rangle\!\rangle_{L^2\otimes^2} &= 0, && \forall \boldsymbol{\Upsilon} \in H^1(T\mathcal{B},\mathcal{S}_1),\\
\langle\!\langle \mathbf{grad}\,\boldsymbol{U},\boldsymbol{\Theta}\rangle\!\rangle_{L^2\otimes^2} + \langle\!\langle \mathbf{curl}^{\mathsf{T}}\boldsymbol{T},\mathbf{curl}^{\mathsf{T}}\boldsymbol{\Theta}\rangle\!\rangle_{L^2\otimes^2} + \langle\!\langle \boldsymbol{P},\boldsymbol{\Theta}\rangle\!\rangle_{L^2\otimes^2} & \\
&= \langle\!\langle \boldsymbol{Q},\boldsymbol{\Theta}\rangle\!\rangle_{L^2\otimes^2}, && \forall \boldsymbol{\Theta} \in H^{\mathbf{c}}(\otimes^2 T\mathcal{B},\mathcal{S}_1),\\
\langle\!\langle \boldsymbol{T},\boldsymbol{\Pi}\rangle\!\rangle_{L^2\otimes^2} &= 0, && \forall \boldsymbol{\Pi} \in \mathcal{H}(\mathcal{B},\mathcal{S}_1,\mathcal{S}_2).
\end{aligned} \tag{4.3}$$

The above equations are the Euler-Lagrange equations corresponding to a saddle point of the functional $\mathscr{I} : H^1(T\mathcal{B},\mathcal{S}_1) \times H^{\mathbf{c}}(\otimes^2 T\mathcal{B},\mathcal{S}_1) \times \mathcal{H}(\mathcal{B},\mathcal{S}_1,\mathcal{S}_2) \to \mathbb{R}$ given by

$$\begin{aligned}
\mathscr{I}(\boldsymbol{\Upsilon},\boldsymbol{\Theta},\boldsymbol{\Pi}) = \frac{1}{2}\langle\!\langle \boldsymbol{\Upsilon},\boldsymbol{\Upsilon}\rangle\!\rangle_{L^2T} - \langle\!\langle \mathbf{grad}\,\boldsymbol{\Upsilon},\boldsymbol{\Theta}\rangle\!\rangle_{L^2\otimes^2} - \frac{1}{2}\langle\!\langle \mathbf{curl}^{\mathsf{T}}\boldsymbol{\Theta},\mathbf{curl}^{\mathsf{T}}\boldsymbol{\Theta}\rangle\!\rangle_{L^2\otimes^2}\\
- \langle\!\langle \boldsymbol{\Pi},\boldsymbol{\Theta}\rangle\!\rangle_{L^2\otimes^2} + \langle\!\langle \boldsymbol{Q},\boldsymbol{\Theta}\rangle\!\rangle_{L^2\otimes^2}.
\end{aligned}$$

The first equation in (4.3) implies that $\boldsymbol{U} = \mathbf{div}_{\mathcal{S}_2}\boldsymbol{T}$. The second equation says that $\boldsymbol{T}$ solves the equation $\mathscr{L}(\boldsymbol{T}) = \boldsymbol{Q} - \boldsymbol{P}$, where the harmonic tensor $\boldsymbol{P}$ is the harmonic part of $\boldsymbol{Q}$ provided by a Hodge-type decomposition. Hence $\boldsymbol{Q} - \boldsymbol{P} \in \mathcal{H}(\mathcal{B},\mathcal{S}_1,\mathcal{S}_2)^\perp$, which is the necessary and sufficient condition for the existence of a solution due to Theorem 4.1. Finally, the third equation eliminates the degree of freedom for choosing a solution $\boldsymbol{T}$ by requiring that $\boldsymbol{T} \in \mathcal{H}(\mathcal{B},\mathcal{S}_1,\mathcal{S}_2)^\perp$. The well-posedness of the mixed formulation (4.3) follows from the Fredholm property of the associated Hilbert complex. The upshot is the following theorem.

Theorem 4.2. *The mixed formulation (4.3) is well-posed. Thus, for any* $\boldsymbol{Q} \in L^2(\otimes^2 T\mathcal{B})$*, the problem (4.3) admits a unique solution* $(\boldsymbol{U},\boldsymbol{T},\boldsymbol{P}) \in H^1(T\mathcal{B},\mathcal{S}_1) \times H^{\mathbf{c}}(\otimes^2 T\mathcal{B},\mathcal{S}_1) \times \mathcal{H}(\mathcal{B},\mathcal{S}_1,\mathcal{S}_2)$*, and there is a constant* $c > 0$ *such that*

$$\|\boldsymbol{U}\|_{H^1T} + \|\boldsymbol{T}\|_{H^{\mathbf{c}}} + \|\boldsymbol{P}\|_{L^2\otimes^2} \leq c\|\boldsymbol{Q}\|_{L^2\otimes^2}.$$

In the mixed formulation (4.3), the boundary conditions on $\mathcal{S}_1$ and $\mathcal{S}_2$ are the essential and the natural boundary conditions, respectively, i.e. the boundary conditions on $\mathcal{S}_1$ are explicitly imposed in the solution spaces while the boundary conditions on $\mathcal{S}_2$ are imposed by the mixed formulation. Alternatively, one can write another mixed formulation of (4.1) as follows: Given $\boldsymbol{Q} \in L^2(\otimes^2 T\mathcal{B})$, find

$(\boldsymbol{S},\boldsymbol{T},\boldsymbol{P}) \in H^{\mathbf{c}}(\otimes^2 T\mathcal{B},\mathcal{S}_2)\times H^{\mathbf{d}}(\otimes^2 T\mathcal{B},\mathcal{S}_2)\times \mathcal{H}(\mathcal{B},\mathcal{S}_1,\mathcal{S}_2)$ such that

$$\begin{aligned}
\langle\!\langle \boldsymbol{S},\boldsymbol{\Sigma}\rangle\!\rangle_{L^2\otimes^2} - \langle\!\langle \boldsymbol{T},\mathbf{curl}^{\mathsf{T}}\boldsymbol{\Sigma}\rangle\!\rangle_{L^2\otimes^2} &= 0, && \forall \boldsymbol{\Sigma}\in H^{\mathbf{c}}(\otimes^2 T\mathcal{B},\mathcal{S}_2),\\
\langle\!\langle \mathbf{curl}^{\mathsf{T}}\boldsymbol{S},\boldsymbol{\Theta}\rangle\!\rangle_{L^2\otimes^2} + \langle\!\langle \mathbf{div}\,\boldsymbol{T},\mathbf{div}\,\boldsymbol{\Theta}\rangle\!\rangle_{L^2T}+\langle\!\langle \boldsymbol{P},\boldsymbol{\Theta}\rangle\!\rangle_{L^2\otimes^2} & \\
&= \langle\!\langle \boldsymbol{Q},\boldsymbol{\Theta}\rangle\!\rangle_{L^2\otimes^2}, && \forall \boldsymbol{\Theta}\in H^{\mathbf{d}}(\otimes^2 T\mathcal{B},\mathcal{S}_2),\\
\langle\!\langle \boldsymbol{T},\boldsymbol{\Pi}\rangle\!\rangle_{L^2\otimes^2} &= 0, && \forall \boldsymbol{\Pi}\in \mathcal{H}(\mathcal{B},\mathcal{S}_1,\mathcal{S}_2).
\end{aligned} \tag{4.4}$$

In this mixed formulation, in contrary to (4.3), the boundary conditions on $\mathcal{S}_2$ are the essential boundary conditions and those on $\mathcal{S}_1$ are the natural boundary conditions.

5 Discrete Hilbert Complexes

It is possible to discretize the Hilbert complex mentioned earlier using some appropriate finite element spaces. Let $\mathcal{B}_h$ be a 3D triangulation. Then, one can write the following polynomial complexes for second-order tensors:

$$0\longrightarrow \mathcal{P}_r^H(T\mathcal{B}_h,\mathcal{S}_1)\xrightarrow{\mathbf{grad}}\mathcal{P}_{r-1}^{\mathbf{c}}(\otimes^2 T\mathcal{B}_h,\mathcal{S}_1)\xrightarrow{\mathbf{curl}^{\mathsf{T}}}\mathcal{P}_{r-2}^{\mathbf{d}}(\otimes^2 T\mathcal{B}_h,\mathcal{S}_1)\xrightarrow{\mathbf{div}}\mathcal{P}_{r-3}(T\mathcal{B}_h)\longrightarrow 0, \tag{5.1a}$$

$$0\longrightarrow \mathcal{P}_r^H(T\mathcal{B}_h,\mathcal{S}_1)\xrightarrow{\mathbf{grad}}\mathcal{P}_{r-1}^{\mathbf{c}}(\otimes^2 T\mathcal{B}_h,\mathcal{S}_1)\xrightarrow{\mathbf{curl}^{\mathsf{T}}}\mathcal{P}_{r-1}^{\mathbf{d}-}(\otimes^2 T\mathcal{B}_h,\mathcal{S}_1)\xrightarrow{\mathbf{div}}\mathcal{P}_{r-2}(T\mathcal{B}_h)\longrightarrow 0, \tag{5.1b}$$

$$0\longrightarrow \mathcal{P}_r^H(T\mathcal{B}_h,\mathcal{S}_1)\xrightarrow{\mathbf{grad}}\mathcal{P}_{r}^{\mathbf{c}-}(\otimes^2 T\mathcal{B}_h,\mathcal{S}_1)\xrightarrow{\mathbf{curl}^{\mathsf{T}}}\mathcal{P}_{r-1}^{\mathbf{d}}(\otimes^2 T\mathcal{B}_h,\mathcal{S}_1)\xrightarrow{\mathbf{div}}\mathcal{P}_{r-2}(T\mathcal{B}_h)\longrightarrow 0, \tag{5.1c}$$

$$0\longrightarrow \mathcal{P}_r^H(T\mathcal{B}_h,\mathcal{S}_1)\xrightarrow{\mathbf{grad}}\mathcal{P}_{r}^{\mathbf{c}-}(\otimes^2 T\mathcal{B}_h,\mathcal{S}_1)\xrightarrow{\mathbf{curl}^{\mathsf{T}}}\mathcal{P}_{r}^{\mathbf{d}-}(\otimes^2 T\mathcal{B}_h,\mathcal{S}_1)\xrightarrow{\mathbf{div}}\mathcal{P}_{r-1}(T\mathcal{B}_h)\longrightarrow 0. \tag{5.1d}$$

Note that (5.1a) is a subcomplex of (5.1b), (5.1b) is a subcomplex of (5.1c), and (5.1c) is a subcomplex of (5.1d). These discrete complexes are Hilbert complexes. Regardless of the refinement level of the mesh $\mathcal{B}_h$, the cohomology groups of the above discrete Hilbert complexes are the same as those of the nonlinear elasticity complex.

6 Stable Mixed Finite Element Methods for the Tensor Laplacian

The bounded cochain projections between the nonlinear elasticity complex and the discrete complexes introduced earlier allow one to use the general theory for approximation of Hilbert complexes for approximating the nonlinear elasticity complexes. Consider the following mixed formulation, which is associated to the first Laplacians of the discrete Hilbert complexes (5.1): Given $\boldsymbol{Q}\in L^2(\otimes^2 T\mathcal{B})$, find $(\boldsymbol{U}_h,\boldsymbol{T}_h,\boldsymbol{P}_h)\in V_h^H\times V_h^{\mathbf{c}}\times\mathcal{H}_h$ such that

$$\begin{aligned}
\langle\!\langle \boldsymbol{U}_h,\boldsymbol{\Upsilon}\rangle\!\rangle_{L^2T} - \langle\!\langle \boldsymbol{T}_h,\mathbf{grad}\,\boldsymbol{\Upsilon}\rangle\!\rangle_{L^2\otimes^2} &= 0, && \forall\boldsymbol{\Upsilon}\in V_h^H,\\
\langle\!\langle \mathbf{grad}\,\boldsymbol{U}_h,\boldsymbol{\Theta}\rangle\!\rangle_{L^2\otimes^2} + \langle\!\langle \mathbf{curl}^{\mathsf{T}}\boldsymbol{T}_h,\mathbf{curl}^{\mathsf{T}}\boldsymbol{\Theta}\rangle\!\rangle_{L^2\otimes^2} + \langle\!\langle \boldsymbol{P}_h,\boldsymbol{\Theta}\rangle\!\rangle_{L^2\otimes^2} &= \langle\!\langle \boldsymbol{Q},\boldsymbol{\Theta}\rangle\!\rangle_{L^2\otimes^2}, && \forall\boldsymbol{\Theta}\in V_h^{\mathbf{c}},\\
\langle\!\langle \boldsymbol{T}_h,\boldsymbol{\Pi}\rangle\!\rangle_{L^2\otimes^2} &= 0, && \forall\boldsymbol{\Pi}\in\mathcal{H}_h,
\end{aligned} \tag{6.1}$$

where the pair $(V_h^H, V_h^{\mathbf{c}})$ can be either

$$\left(\mathcal{P}_{r+1}^H(T\mathcal{B}_h,\mathcal{S}_1),\mathcal{P}_r^{\mathbf{c}}(\otimes^2 T\mathcal{B}_h,\mathcal{S}_1)\right) \text{ or } \left(\mathcal{P}_{r}^H(T\mathcal{B}_h,\mathcal{S}_1),\mathcal{P}_r^{\mathbf{c}-}(\otimes^2 T\mathcal{B}_h,\mathcal{S}_1)\right),$$

with $r \geq 1$. Also depending on the choice of $(V_h^H, V_h^{\mathbf{c}})$, $\mathcal{H}_h := \ker \mathbf{curl}_{\mathcal{S}_1}^{\mathsf{T}} \cap \ker \mathbf{grad}_h^*$, is the space of discrete harmonic tensors associated to one of the discrete Hilbert complexes (5.1). Clearly, we have $V_h^H \subset H^1(T\mathcal{B}, \mathcal{S}_1)$ and $V_h^{\mathbf{c}} \subset H^{\mathbf{c}}(\otimes^2 T\mathcal{B}, \mathcal{S}_1)$. However, $\mathcal{H}_h \notin \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$, in general. Therefore, the discrete mixed formulation (6.1) is a generalized (or non-conformal) Galerkin method for the mixed formulation (4.3).

The Fredholm property of the original Hilbert complex and the existence of the bounded cochain projections imply that for a shape regular family of triangulations $\{\mathcal{B}_h\}$, the mixed finite element methods based on (6.1) are stable and convergent. The rate of convergence is determined by the smoothness of the data $\boldsymbol{Q}$, the smoothness of the domain, and the degrees of polynomials that generate the finite element spaces. One can also show that the error is the optimal order allowed by discrete solution spaces if there is sufficient elliptic regularity. For example, let $(V_h^H, V_h^{\mathbf{c}}) = \big(\mathcal{P}_{r+1}^H(T\mathcal{B}_h, \mathcal{S}_1), \mathcal{P}_r^{\mathbf{c}}(\otimes^2 T\mathcal{B}_h, \mathcal{S}_1)\big)$, and suppose that the solutions $\boldsymbol{T}$ and $\boldsymbol{P}$ are of Sobolev class H^2. Then, one can write

$$\begin{aligned}\|\boldsymbol{U} - \boldsymbol{U}_h\|_{L^2 T} + h\|\mathbf{grad}(\boldsymbol{U} - \boldsymbol{U}_h)\|_{L^2 \otimes^2} + h\|\boldsymbol{T} - \boldsymbol{T}_h\|_{L^2\otimes^2} \\ + h^2\|\mathbf{curl}^{\mathsf{T}}(\boldsymbol{T} - \boldsymbol{T}_h)\|_{L^2\otimes^2} + h\|\boldsymbol{P} - \boldsymbol{P}_h\|_{L^2\otimes^2} = O(h^{r+2}).\end{aligned}$$

Therefore, regarding the degree of the polynomial approximation, all components converge with the optimal order.

For implementing (6.1), one needs to calculate the space of discrete harmonic tensors $\mathcal{H}_h$. To this end, consider the following homogeneous problem: Find $(\boldsymbol{U}_h, \boldsymbol{T}_h) \in V_h^H \times V_h^{\mathbf{c}}$ such that

$$\begin{aligned}\langle\!\langle \boldsymbol{U}_h, \boldsymbol{\Upsilon} \rangle\!\rangle_{L^2 T} - \langle\!\langle \boldsymbol{T}_h, \mathbf{grad}\, \boldsymbol{\Upsilon} \rangle\!\rangle_{L^2\otimes^2} = 0, \quad \forall \boldsymbol{\Upsilon} \in V_h^H, \\ \langle\!\langle \mathbf{grad}\, \boldsymbol{U}_h, \boldsymbol{\Theta} \rangle\!\rangle_{L^2\otimes^2} + \langle\!\langle \mathbf{curl}^{\mathsf{T}} \boldsymbol{T}_h, \mathbf{curl}^{\mathsf{T}} \boldsymbol{\Theta} \rangle\!\rangle_{L^2\otimes^2} = 0, \quad \forall \boldsymbol{\Theta} \in V_h^{\mathbf{c}}.\end{aligned}$$

Then, $(\boldsymbol{U}_h, \boldsymbol{T}_h)$ is a solution if and only if $\boldsymbol{U}_h = 0$, and $\boldsymbol{T}_h \in \mathcal{H}_h$.

7 Acknowledgment/Disclaimer

This work was sponsored (in part) by the Air Force Office of Scientific Research, USAF, under grant number FA9550-10-1-0378. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Air Force Office of Scientific Research or the U.S. Government.

8 Personal Supported During Duration of Grant

Arzhang Angoshtari Graduate Student, Georgia Institute of Technology, Atlanta

9 Publications

1) A. Angoshtari and A. Yavari, Stable mixed finite element methods for the tensor Laplacian with mixed boundary conditions, *In preparation.*

2) A. Angoshtari and A. Yavari, On the compatibility equations of nonlinear and linear elasticity in the presence of boundary conditions, *Submitted to ZAMP.*

3) A. Angoshtari and A. Yavari, The weak compatibility equations of nonlinear elasticity and the insufficiency of the Hadamard jump condition for non-simply connected bodies, *Submitted to Continuum Mechanics and Thermodynamics.*

4) A. Angoshtari and A. Yavari, Hilbert complexes, orthogonal decompositions, and potentials for nonlinear continua, *Submitted to Journal of Mathematical Physics.*

5) A. Angoshtari and A. Yavari, Differential complexes in continuum mechanics, *Archive for Rational Mechanics and Analysis*, **216**, 193-220, 2015.

6) A. Angoshtari and A. Yavari, A geometric structure-preserving discretization scheme for incompressible linearized elasticity, *Computer Methods in Applied Mechanics and Engineering*, **259**, 130-153, 2013.

10 Honors and Awards Received

None

11 AFRL Point of Contact

None

12 Transitions

None

13 New Discoveries

None

1.

1. Report Type

F na Report

Primary Contact E-mail
Contact email if there is a problem with the report.

arash.yavar @ce.gatech.edu

Primary Contact Phone Number
Contact phone number if there is a problem with the report

4048942436

Organization / Institution name

Georg a Inst tute of Techno ogy

Grant/Contract Title
The full title of the funded effort.

GEOMETRIC STRUCTURE-PRESERVING DISCRETIZATION SCHEMES FOR ELASTICITY

Grant/Contract Number
AFOSR assigned control number. It must begin with "FA9550" or "F49620" or "FA2386".

FA9550-12-1-0290

Principal Investigator Name
The full name of the principal investigator on the grant or contract.

Arash Yavar

Program Manager
The AFOSR Program Manager currently assigned to the award

Jean-Luc Camb er

Reporting Period Start Date

05/31/2012

Reporting Period End Date

05/31/2015

Abstract

We ntroduced a smooth comp ex for non near e ast c ty that can be cons dered as the tensor a ana ogue of the standard grad-cur -d v comp ex. Th s mathemat ca structure s mu taneous y descr bes the k nemat cs and the k net cs of arge deformat ons. The re at on between th s comp ex and the de Rham comp ex a ows one to read y der ve the necessary and suff c ent cond t ons for the compat b   ty of d sp acement grad ent and the ex stence of stress funct ons on non-contract b e bod es. The ma n app cat on of the non near e ast c ty comp ex s n deve op ng m xed f n te e ement methods for arge deformat ons, wh ch w   be pursued n a future project. To th s end, the smooth comp ex shou d be extended to a so nc ude ess smooth tensors. We ntroduced th s extens on by us ng the so-ca ed part y Sobo ev spaces. The resu t s a H bert comp ex nvo v ng second-order tensors on f at compact man fo ds w th boundary. We then used the genera framework of H bert comp exes to wr te Hodge-type and He mho tz-type orthogona decompos t ons for second-order tensors. As some app cat ons of these decompos t ons n cont nuum mechan cs, one can study the stra n compat b   ty equat ons of non near e ast c ty n the presence of D r ch et boundary cond t ons.

Distribution Statement
This is block 12 on the SF298 form.

D str but on A - Approved for Pub c Re ease

Explanation for Distribution Statement

If this is not approved for public release, please provide a short explanation. E.g., contains proprietary information.

SF298 Form

Please attach your SF298 form. A blank SF298 can be found here. Please do not password protect or secure the PDF The maximum file size for an SF298 is 50MB.

AFD-070820-035 Yavar SF298.pdf

Upload the Report Document. File must be a PDF. Please do not password protect or secure the PDF . The maximum file size for the Report Document is 50MB.

Report_AFOSR_Yavar .pdf

Upload a Report Document, if any. The maximum file size for the Report Document is 50MB.

Archival Publications (published) during reporting period:

A. Angoshtar and A. Yavar , D fferent a comp exes n cont nuum mechan cs, Arch ve for Rat ona Mechan cs and Ana ys s, 216, 193-220, 2015.

A. Angoshtar and A. Yavar , A geometr c structure-preserv ng d scret zat on scheme for ncompress b e near zed e ast c ty, Computer Methods n App ed Mechan cs and Eng neer ng, 259, 130-153, 2013.

Changes in research objectives (if any):

Change in AFOSR Program Manager, if any:

The program manager used to be Dr. Far ba Fahroo.

Extensions granted or milestones slipped, if any:

AFOSR LRIR Number

LRIR Title

Reporting Period

Laboratory Task Manager

Program Officer

Research Objectives

Technical Summary

Funding Summary by Cost Category (by FY, $K)

	Start ng FY	FY+1	FY+2
Sa ary			
Equ pment/Fac t es			
Supp es			
Tota			

Report Document

Report Document - Text Analysis

Report Document - Text Analysis

Appendix Documents

2. Thank You

E-mail user

Aug 04, 2015 13:53:04 Success: Ema Sent to: arash.yavar @ce.gatech.edu